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Scattering on fractal measures

Charles-Antoine Guerin† and Matthias Holschneider‡
Centre de Physique Théorique§, CNRS, Luminy, Case 907 F-13288 Marseille Cedex 9, France

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Abstract. We study the one-dimensional potential-scattering problem when the potential is a Radon measure with compact support. We show that the usual reflection and transmission amplitude $r(p)$ and $t(p)$ of an incoming wave e^{ipx} are well defined. We also show that the scattering problem on fractal potentials can be obtained as a limit case of scattering on smooth potentials. We then explain how to retrieve the fractal 2-wavelet dimension and/or the correlation dimension of the potential by means of the reflexion amplitude $r(p)$. We study the particular case of self-similar measures and show that, under some general conditions, $r(p)$ has a large-scale renormalization. A numerical application is presented.

1. Introduction

Scattering on fractal systems has been extensively studied during the last decade, because it provides a powerful tool to characterize irregular surfaces or volumes. Consider an incoming wave arriving on an obstacle with wave vector $\boldsymbol{p}_{\text{in}}$ and look at the scattered intensity $I(\boldsymbol{p}_{\text{in}}, \boldsymbol{p}_{\text{out}})$ in the direction $\boldsymbol{p}_{\text{out}}$. It appears that $I(\boldsymbol{p}_{\text{in}}, \boldsymbol{p}_{\text{out}}) = I(\boldsymbol{q})$, where $\boldsymbol{q} = \boldsymbol{p}_{\text{in}} - \boldsymbol{p}_{\text{out}}$ is the momentum transfer. The scattered intensity $I(\boldsymbol{q})$ is then usually connected to the fractal properties of the obstacle. The most interesting result is the so-called power-law scattering, which occurs at small-angle x-ray or neutron scattering. Rough materials are well modelled by random fractals and the scattered intensity on such structures appears to scale with some power of $q = |\boldsymbol{q}|$:

$$I(\boldsymbol{q}) \sim q^{-D} \tag{1.1}$$

where the exponent D is a fractal dimension of the system, depending on whether it is a mass fractal, a surface fractal or a pore fractal (good surveys can be found in [Pfe85], [Pfe88] or [Sin89]).

Surprisingly, things happen to be more complicated when one deals with non-random fractals. Schmidt and Dacai [SD86] performed small-angle scattering on the Menger sponge and observed some complicated behaviour of the scattered intensity which did not match with a power law. Similarly, Allain and Cloitre [AC85] investigated the optical diffraction on a deterministic fractal grating. By use of Fresnel's formula, they computed the intensity scattered by a bidimensional Cantor-like grating illuminated by a converging spherical wave. They found that power-law scattering held in the averaged sense only:

$$\int_{q/3}^{q} dq' \, I(q') \sim q^{1-D}$$

† E-mail address: guerin@cpt.univ-mrs.fr
‡ E-mail address: hols@cpt.univ-mrs.fr
§ Unité Propre de Recherche 7061.

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where $D = \log 2/\log 3$ is the fractal dimension of the grating. Now look at the backscattered intensity $I(q)$ (figure 1) on a potential barrier in one dimension, where the potential is again a Cantor-like feature (this will be made clearer in the following). The slope given by the local maxima of the graph is -2, which is a trivial decrease specific to dimension one and has nothing to do with the internal structure of the scattering potential. The 'true' power law can only be observed after some suitable averages have been performed (figure 2). Thus, it seems that the relevant quantity to look at is the integrated scattered intensity rather than the scattered intensity itself. The classical result 1.1 may be only an artefact of randomness.

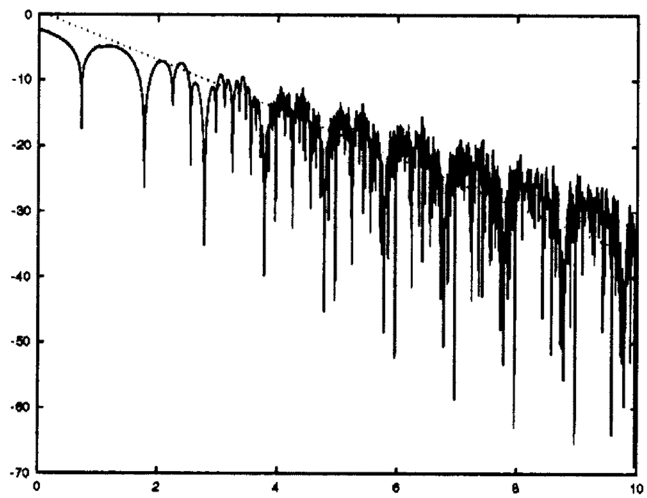

Figure 1. $I(q)$ in log–log diagram for the triadic Cantor measure.

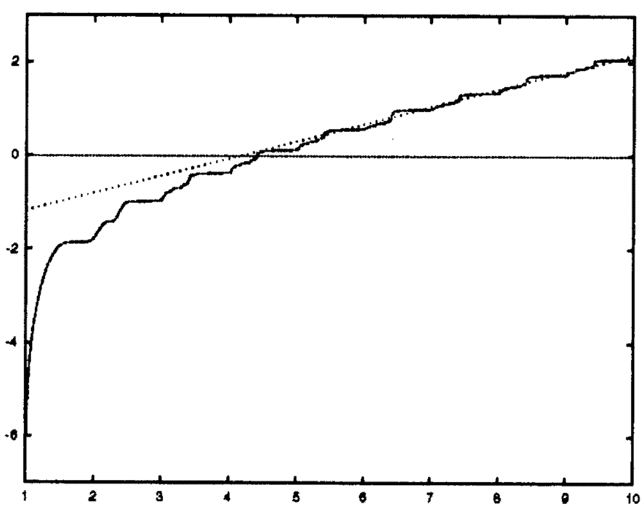

Figure 2. $\int_1^q \mathrm{d}p' p'^2 I(p')$ in log–log diagram. The slope $0.37 \simeq 1 - \log 2/\log 3$ gives the fractal dimension of the potential.

In this paper, we wish to make a detailed study of scattering on deterministic fractals in the simple framework of one-dimensional potential scattering. Since we want to deal with potentials of very low regularity, we assume that the potential V is not a function anymore, but merely a measure. We show that a scattering problem can still be defined in this case. We then retrieve some fractal properties of the potentials by means of the averaged scattering data. In the particular case of self-similar potentials, we also show that the scattering data admit a renormalization procedure linking the different energy scales. The last section is devoted to a numerical illustration of the above results.

2. Scattering on measures

2.1. Scattering formalism in one dimension

Consider the one-dimensional Helmholtz equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p^2\right)\Psi_{V,p} = V\Psi_{V,p} \tag{2.1}$$

where V is a multiplication operator by some smooth real-valued function $V(x)$ with compact support. By 'smooth' we mean that $V(x)$ is of regularity C^k for some $k \geqslant 0$, that is k times continuously differentiable on $\mathbb{R}$. Left of the support of V, the solution $\Psi_{V,p}$ of (2.1) has the form $\Psi_{V,p}(x) = A\,\mathrm{e}^{\mathrm{i}px} + B\,\mathrm{e}^{-\mathrm{i}px}$. Right of the support, it has the same form with other coefficients, say $\Psi_{V,p}(x) = A'\mathrm{e}^{\mathrm{i}px} + B'\mathrm{e}^{-\mathrm{i}px}$. The complex coefficients A, B, A' and B' are related by the so-called transfer matrix $\mathbf{M}(V,p)$:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{M}(V,p)\begin{pmatrix} A' \\ B' \end{pmatrix}.$$

For each p, there are two complex numbers $t_V(p)$ and $r_V(p)$ uniquely defined by

$$\begin{pmatrix} 1 \\ r_V(p) \end{pmatrix} = \mathbf{M}(V,p)\begin{pmatrix} t_V(p) \\ 0 \end{pmatrix}.$$

The coefficients $t_V(p)$ and $r_V(p)$ are known as the transmission and reflexion amplitude, respectively. They represent the specific response of the obstacle V to an ingoing wave $\mathrm{e}^{+\mathrm{i}px}$, and satisfy the energy conservation $|t_V(p)|^2 + |r_V(p)|^2 = 1$. When no confusion is possible, we shall omit in the following the dependance on V and simply write $t(p)$ or $r(p)$.

It can be shown that the Helmholtz equation (2.1) together with boundary conditions

$$\Psi_{V,p}(x) = \begin{cases} \mathrm{e}^{+\mathrm{i}px} + r(p)\,\mathrm{e}^{-\mathrm{i}px} & x \to -\infty \\ t(p)\,\mathrm{e}^{+\mathrm{i}px} & x \to +\infty \end{cases} \tag{2.2}$$

is equivalent to the Lippmann–Schwinger integral equation:

$$\Psi_{V,p}(x) = \mathrm{e}^{\mathrm{i}px} + \int \mathrm{d}y\,\frac{\mathrm{e}^{\mathrm{i}p|x-y|}}{2\mathrm{i}p}V(y)\Psi_{V,p}(y). \tag{2.3}$$

We wish to show that this equation can be solved in the more general framework of potentials defined as measures. Therefore, let us introduce $C^0(\mathbb{R})$ the Banach space of complex-valued continuous functions in $\mathbb{R}$ endowed with the norm $\|\phi\|_\infty = \sup_{x\in\mathbb{R}}|\phi(x)|$. Let $\mathcal{M}$ be the dual of $C^0(\mathbb{R})$, the space of Radon measures. A strong topology on $\mathcal{M}$ may be defined via the norm

$$\|\mu\| = \sup_{\phi\in C^0(\mathbb{R})\setminus\{0\}} \frac{|\int\phi\,\mathrm{d}\mu|}{\|\phi\|_\infty} = \int|\,\mathrm{d}\mu|.$$

The weak topology of $\mathcal{M}$ is induced by the linear functionals $\mu \to \langle \mu, \phi \rangle \equiv \int \phi \, \mathrm{d}\mu$ for $\phi \in C^0(\mathbb{R})$. We say μ is real if

$$\overline{\mu(\phi)} = \mu(\bar{\phi}).$$

Finally we say μ is positive if $\phi \geqslant 0$ implies $\mu(\phi) \geqslant 0$.

Now suppose V is a Radon measure with compact support K_V. The Helmholtz equation (2.1) does not make sense as it stands. However, the Lippmann–Schwinger equation still holds if we agree that $V(x)$ is the distribution such that

$$\int \mathrm{d}x \, \phi(x) V(x) = \int \phi \, \mathrm{d}V.$$

The Lippmann–Schwinger equation can be more clearly rewritten in an operator form

$$\Psi_{V,p} = \Phi_p + G_p * (V \Psi_{V,p}) \tag{2.4}$$

where we have set $G_p(x) = (1/2\mathrm{i}p) \, \mathrm{e}^{+\mathrm{i}p|x|}$ and $\Phi_p(x) = \mathrm{e}^{+\mathrm{i}px}$, $*$ standing for the convolution. The multiplication of a measure by a continuous function is a measure. The convolution of a finite measure with a continuous function is again a continuous function. Thus, it makes sense to seek a solution $\Psi_{V,p}$ of (2.4) in $C^0(\mathbb{R})$. The solution actually exists and is unique. Indeed, consider the following operator from $C^0(\mathbb{R})$ into itself:

$$\mathcal{A}_{V,p} : \phi \mapsto G_p * (V\phi).$$

Since $\|\mathcal{A}_{V,p}\| \leqslant \|V\|/2p$, the operator $(1 - \mathcal{A}_{V,p})$ is invertible as soon as $p > \|V\|/2$, in which case there is a unique continuous function

$$\Psi_{V,p} = (1 - \mathcal{A}_{V,p})^{-1} \Phi_p$$

satisfying (2.4). This function is given explicitly by the Neumann series

$$\Psi_{V,p} = \sum_{n=0}^{\infty} \mathcal{A}_{V,p}^n \Phi_p$$

which converges uniformly in $C^0(\mathbb{R})$. If we let $x \to +\infty$ in (2.3) and identify with (2.2), we obtain integral expressions for the reflection and transmission amplitude:

$$r(p) = \frac{1}{2\mathrm{i}p} \int \mathrm{d}y \, \mathrm{e}^{\mathrm{i}py} \Psi_{V,p}(y) V(y)$$

$$t(p) = 1 + \frac{1}{2\mathrm{i}p} \int \mathrm{d}y \, \mathrm{e}^{-\mathrm{i}py} \Psi_{V,p}(y) V(y). \tag{2.5}$$

Therefore, for a compactly supported Radon measure V the scattering problem has a well-defined meaning since there exists continuous functions $\Psi_{V,p}$, $r_V(p)$ and $t_V(p)$ as soon as $p > \|V\|/2$. Of course, since any smooth function with compact support can be identified with a measure, this also holds for smooth potentials. Note that, in this case, the solution is actually of higher regularity: if V is C^k, we know by Weyl's lemma (see e.g [RS]) that $\Psi_{V,p}$ is at least C^{k+1}.

We are now going to show that the scattering problem on irregular potentials can be obtained as limiting case of scattering on smooth potentials.

First we need the following.

Lemma 2.1. Let $\lambda > 0$ and define $\mathcal{M}_{\lambda,K}$ as the subset of Radon measures with norm less than λ and support contained in K. Then the three (nonlinear) operators

$$\begin{array}{lll} \mathcal{M}_{\lambda,K} \to C^0(\mathbb{R}) & \mathcal{M}_{\lambda,K} \to \mathbb{C} & \mathcal{M}_{\lambda,K} \to \mathbb{C} \\ V \mapsto \Psi_{V,p} & V \mapsto r_V(p) & V \mapsto t_V(p) \end{array}$$

are compact for any $p > \lambda/2$, in the sense that they map weakly convergent sequences into strongly convergent sequences.

As an immediate consequence we have

Proposition 2.1. For all V in $\mathcal{M}$, there exists a sequence of compactly supported smooth functions $V_n \in C_0^\infty$, such that, for all $p > \|V\|/2$,

$$s - \lim_{n\to\infty} \Psi_{V_n,p} = \Psi_{V,p} \qquad \lim_{n\to\infty} r_{V_n}(p) = r_V(p) \qquad \lim_{n\to\infty} t_{V_n}(p) = t_V(p).$$

Proof. Let h be a compactly supported C^∞ function such that $0 \leqslant h \leqslant 1$ and $\int h = 1$, and set $h_n(x) = nh(nx)$. The sequence h_n is an approximate identity and $V_n = h_n * V$ converges to V in the weak topology of $\mathcal{M}$. Moreover, V_n is uniformly bounded by $\|V\|$. Hence, the direct application of lemma (2.1) states the proof. □

An important consequence is that the conservation of energy still holds for potentials defined as measures:

$$\forall p > \|V\|/2 \qquad |r_V(p)|^2 + |t_V(p)|^2 = 1.$$

Indeed, this identity holds for smooth potentials and by the proposition we may go to the limit.

Proof of lemma 2.1. Take some V in $\mathcal{M}_{\lambda,K}$ and a weakly convergent sequence $V_n \to V$ in $\mathcal{M}_{\lambda,K}$. We first show that $\|\chi_K(\Psi_{V_n,p} - \Psi_{V,p})\|_\infty \to 0$, where χ_K is the characteristic function of K. Set $f^j = \chi_K \mathcal{A}^j_{V,p}\Phi_p$ and $f^j_n = \chi_K \mathcal{A}^j_{V_n,p}\Phi_p$. Then

$$\|\chi_K(\Psi_{V_n,p} - \Psi_{V,p})\|_\infty = \left\| \sum_{j=0}^{\infty} (f^j_n - f^j) \right\|_\infty \leqslant \sum_{j=0}^{N} \|f^j_n - f^j\|_\infty + 2 \sum_{j=N+1}^{\infty} \left(\frac{\lambda}{2p}\right)^j.$$

By taking N to infinity, we can make the second term on the right-hand side as small as we want, independently of n. Thus, it suffices to prove the vanishing of each term $\|f^j_n - f^j\|_\infty$ separately when $n \to \infty$. This is clearly the case of $j = 0$. Now we proceed by recurrence. Assume $\|f^j_n - f^j\|_\infty \to 0$ for some $j \geqslant 1$. Then, noting that $f^{j+1} = \chi_K \mathcal{A}_{V,p} f^j$ and $f^{j+1}_n = \chi_K \mathcal{A}_{V,p} f^j_n$, we have

$$\begin{aligned} |f^{j+1}_n - f^{j+1}| &\leqslant \chi_K |\mathcal{A}_{V_n,p}(f^j_n - f^j)| + \chi_K |(\mathcal{A}_{V_n,p} - \mathcal{A}_{V,p}) f^j| \\ &\leqslant \lambda/2p \|f^j_n - f^j\|_\infty + \chi_K |G_p * (V_n - V) f^j|. \end{aligned}$$

By hypothesis, the sequence of functions $\chi_K(x)|G_p * (V_n - V) f^j(x)|$ converges pointwise to zero. Since all these functions have their support on the same compact K, the convergence is uniform. Thus $\|f^{j+1}_n - f^{j+1}\|_\infty \to 0$ and the recurrence is proved.

Then, since $p > \lambda/2$,

$$\begin{aligned} r_{V_n}(p) - r_V(p) &= \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}px} V_n(x)(\Psi_{V_n,p}(x) - \Psi_{V,p}(x)) \\ &\quad + \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}px} (V_n(x) - V(x))\Psi_{V,p}(x) \end{aligned}$$

and we have

$$\lim_{n\to\infty} |r_{V_n}(p) - r_V(p)| \leqslant \lim_{n\to\infty} \lambda \|\chi_K(\Psi_{V_n,p} - \Psi_{V,p})\|_\infty + \lim_{n\to\infty} |\langle V_n - V, \Phi_p \Psi_{V,p}\rangle| = 0$$

and similarly for $|t_{V_n}(p) - t_V(p)|$. Finally, since by (2.2),

$$\|(1 - \chi_K)(\Psi_{V_n,p} - \Psi_{V,p})\|_\infty \leqslant \max\{|r_{V_n}(p) - r_V(p)|, |t_{V_n}(p) - t_V(p)|\}$$

we also have $\lim_{n\to\infty} \|\Psi_{V_n,p} - \Psi_{V,p}\|_\infty = 0$. This completes the proof. □

Now let us estimate the reflection amplitude. In view of (2.5), $r(p)$ can be written as a series

$$r(p) = \sum_{n=1}^{\infty} \rho_n(p)$$

with

$$\rho_n(p) = \frac{1}{2\mathrm{i}p} \int \mathrm{d}y\, \Phi_p(y) \mathcal{A}_{V,p}^{n-1} \Phi_p(y).$$

The first term $\rho_1(p)$ is simply given by

$$\rho_1(p) = \frac{1}{2\mathrm{i}p} \hat{V}(-2p) \tag{2.6}$$

where $\hat{V}$ is the Fourier transform of V:

$$\hat{V}(\xi) = \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}\xi x} V(x).$$

$\rho_1(p)$ is called the Born approximation for the reflection amplitude. The contribution of the higher-order terms can be estimated. A straightforward calculation shows that

$$\left| \sum_{n=1}^{\infty} \rho_n(p) \right| \leqslant O\left(\frac{1}{p^2}\right)$$

that is

$$r(p) = \frac{\hat{V}(-2p)}{2\mathrm{i}p} + O\left(\frac{1}{p^2}\right). \tag{2.7}$$

Recall that the notation $f = O(g)$ means $\forall x,\ f(x) \leqslant Cg(x)$ for some positive constant C.

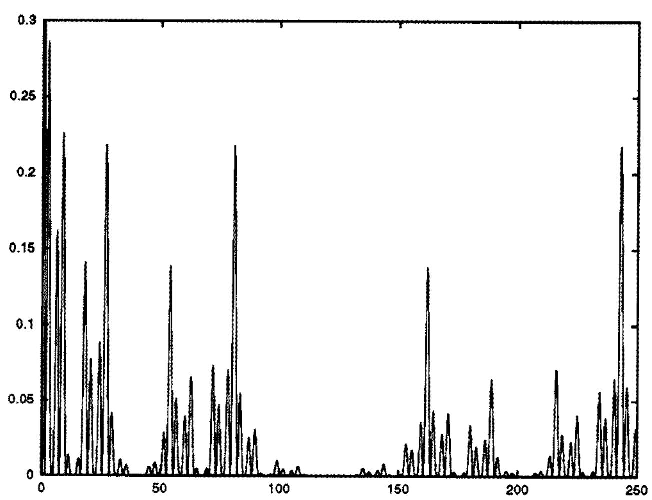

Figure 3. $|\hat{V}(p)|^2$ for V the triadic Cantor measure.

2.2. Recovering some fractal dimensions

As we have seen, the reflection and transmission amplitude are closely linked to the Fourier transform of the potential. For smooth potentials, the Fourier transform vanishes at high frequencies and so scattering data have a trivial asymptotic behaviour. However, for some singular potentials (see e.g figure 3), the Fourier transform never ceases to oscillate. This induces an apparently chaotic behaviour in the scattering data.

However, the $\mathcal{L}^2$-Fourier asymptotic of finite measures follow a general drift, which is governed by their fractal 2-wavelet dimension [Hol94a]. For positive measures, the 2-wavelet dimension reduces to the classical correlation dimension, which is perhaps better known. One of the usual definitions of this last dimension is the following [Pes93].

For any positive measure V, define the function

$$\Omega_V(r) = \int \mathrm{d}V(x)V(B(x,r))$$

where $B(x,r)$ is the ball of radius r around the point x. The quantities

$$D^+(V) = \limsup_{r\to 0} \frac{\log \Omega_V(r)}{\log r} \qquad D^-(V) = \liminf_{r\to 0} \frac{\log \Omega_V(r)}{\log r}$$

are called the upper and lower correlation dimension of V, respectively. When $D^-(V)$ and $D^+(V)$ coincide, we write $D(V)$ for their common value and call it simply the correlation dimension. Heuristically, these dimensions indicate the rate of correlation of the measure. Indeed, suppose V is a probability measure. Then, if you pick randomly two points x_1 and x_2 according to the law V, the probability that they are closer than ϵ is given by

$$\begin{aligned}\mathrm{Proba}(|x_1 - x_2| < r) &= \int \mathrm{d}V\,(x)\,\mathrm{d}V\,(y)\,1_{|x-y|<r} \\ &= \int \mathrm{d}V(x)\,\mathrm{d}V(y)\,1_{y\in B(x,r)} \\ &= \int \mathrm{d}V(x)V(B(x,r)) \\ &= \Omega_V(r).\end{aligned}$$

For positive finite measures, it was shown in [GH95a] that the $\mathcal{L}^2$-Fourier asymptotic is governed by the correlation dimensions. Precisely, we have

Lemma 2.2. ([GH95a]) For any finite positive Radon measure V on $\mathbb{R}$,

$$\limsup_{p\to\infty} \frac{\log(\int_{0\leqslant\xi\leqslant p} \mathrm{d}\xi\,|\hat{V}(\xi)|^2)}{\log p} = 1 - D^-(V)$$

$$\liminf_{p\to\infty} \frac{\log(\int_{0\leqslant\xi\leqslant p} \mathrm{d}\xi\,|\hat{V}(\xi)|^2)}{\log p} = 1 - D^+(V).$$

For signed measures, the correlation dimensions are not sufficient to characterize the Fourier asymptotic. The relevant dimensions in this case are the upper and lower wavelet dimensions $\kappa_2^{\pm}$ introduced in [Hol94a]. They are defined via the following procedure: take some function g in $\mathcal{S}^+$, that is a function in the Schwartz class whose Fourier transform is supported by positive frequencies only, and consider the wavelet transform of V with respect to g,

$$\mathcal{W}_g V(b,a) = (\tilde{g}_a * V)(b)$$

where $\tilde{g}_a(x) = a^{-1}\bar{g}(-a^{-1}x)$ is essentially a dilated version of g. Now set

$$G_g(a) = \int \mathrm{d}b |\mathcal{W}_g V(b,a)|^2$$

and

$$\Gamma_g(a) = \min\left\{ \int_a^1 \frac{\mathrm{d}\alpha}{\alpha} G_g(\alpha), \int_0^a \frac{\mathrm{d}\alpha}{\alpha} G_g(\alpha) \right\}.$$

Then the 2-wavelet dimensions $\kappa_2^{\pm}(V)$ are defined as

$$\kappa_2^+(V) = \limsup_{a\to 0} \frac{\log \Gamma_g(a)}{\log a} \qquad \kappa_2^-(V) = \liminf_{a\to 0} \frac{\log \Gamma_g(a)}{\log a}$$

and do not depend on the chosen function g in $\mathcal{S}^+$ provided $g \neq 0$. They satisfy the following

Lemma 2.3. ([Hol94a]) If V is a finite complex Radon measure, then

$$V \notin \mathcal{L}^2(\mathbb{R}) \qquad \text{and} \qquad \begin{cases} \displaystyle\limsup_{p\to\infty} \frac{\log(\int_{0\leqslant\xi\leqslant p} \mathrm{d}\xi |\hat{V}(\xi)|^2)}{\log p} = -\kappa_2^-(V) \\ \displaystyle\liminf_{p\to\infty} \frac{\log(\int_{0\leqslant\xi\leqslant p} \mathrm{d}\xi |\hat{V}(\xi)|^2)}{\log p} = -\kappa_2^+(V) \end{cases}$$

or

$$V \in \mathcal{L}^2(\mathbb{R}) \qquad \text{and} \qquad \begin{cases} \displaystyle\limsup_{p\to\infty} \frac{\log(\int_{\xi\geqslant p} \mathrm{d}\xi |\hat{V}(\xi)|^2)}{\log p} = -\kappa_2^-(V) \\ \displaystyle\liminf_{p\to\infty} \frac{\log(\int_{\xi\geqslant p} \mathrm{d}\xi |\hat{V}(\xi)|^2)}{\log p} = -\kappa_2^+(V). \end{cases}$$

The next theorem is a direct application of lemma 2.3.

Theorem 2.4. Let V be a finite, real Radon measure with compact support and let p_0 be some real number greater than $\|V\|/2$. Then either $V \notin \mathcal{L}^2(\mathbb{R})$ and

$$\begin{cases} \displaystyle\limsup_{p\to\infty} \frac{\log \int_{p_0}^p \mathrm{d}p'\, p'^2 |r(p')|^2}{\log p} = -\kappa_2^-(V) \qquad (= 1 - D^-(V) \text{ if } V \geqslant 0) \\ \displaystyle\liminf_{p\to\infty} \frac{\log \int_{p_0}^p \mathrm{d}p'\, p'^2 |r(p')|^2}{\log p} = -\kappa_2^+(V) \qquad (= 1 - D^+(V) \text{ if } V \geqslant 0) \end{cases}$$

or $V \in \mathcal{L}^2(\mathbb{R})$ and the previous limits are zero.

Proof. The proof is elementary but we give it anyway for the convenience of the reader.

Take some $p_0 > \|V\|/2$ and define the functions

$$I(p) = \int_{p_0}^p \mathrm{d}p'\, p'^2 |r(p')|^2$$

$$J(p) = \int_{p_0}^p \mathrm{d}\xi |\hat{V}(\xi)|^2.$$

It suffices to show that

$$\limsup_{p\to\infty} \frac{\log I(p)}{\log p} = \limsup_{p\to\infty} \frac{\log J(p)}{\log p} \qquad \liminf_{p\to\infty} \frac{\log I(p)}{\log p} = \liminf_{p\to\infty} \frac{\log J(p)}{\log p}. \tag{2.8}$$

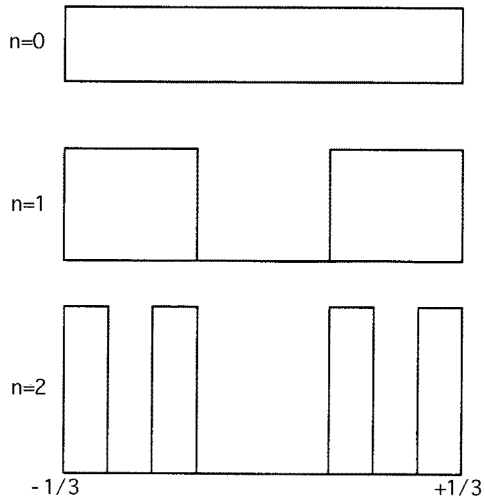

Figure 4. The first steps of the construction of a Cantor measure.

From equation (2.7), we have

$$I(p) = J(p) + O(\log p).$$

Now $\log(x + y) = \log x + O(y/x)$ and thus

$$\frac{\log I(p)}{\log p} = \frac{\log I(p)}{\log p} + O\left(\frac{1}{J(p)}\right). \tag{2.9}$$

Since $J(p)$ is an increasing function, it either converges to some positive constant or goes to infinity. If it converges, then $I(p) = O(\log p)$ and

$$\lim_{p\to\infty} \frac{\log I(p)}{\log p} = \lim_{p\to\infty} \frac{\log J(p)}{\log p} = 0.$$

If $J(p)$ diverges, then (2.8) follows from (2.9). □

3. Scattering on self-similar measures

We now consider the case of a potential given by a self-similar measure. A self-similar measure on $\mathbb{R}$ was defined by Hutchinson [Hut81] to be a probability measure V satisfying

Condition 3.A. (Self-similarity equation.) $V = \sum_{i=1}^{N} p_i V \circ S_i^{-1}$

where the p_i are a set of weights ($0 < p_i < 1$, $\sum_{i=1}^{N} p_i = 1$) and the S_i a set of similarities $S_i(x) = l_i x + b_i$, with $0 < l_i < 1$. The support of V is the unique non-empty compact K globally invariant under these contractions, i.e. $K = \cup_{i=1}^{N} S_i(K)$. This support may be of zero Lebesgue measure. This occurs when the following condition is satisfied.

Condition 3.B. (Strong open set condition [Str93].) There exists a closed bounded set U such that $S_i(U) \cap S_j(U) = 0$ for $i \neq j$.

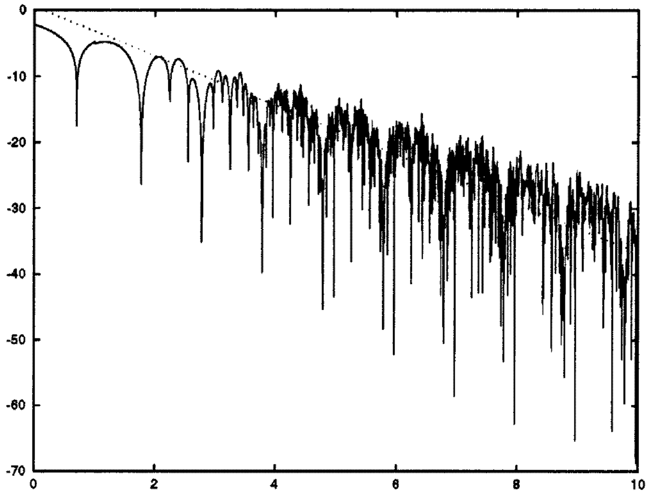

Figure 5. $\log_3 |r(p)|^2$ against $\log_3(p)$ for the standard triadic Cantor measure ($p_1 = p_2 = \frac{1}{2}, l_1 = l_2 = \frac{1}{3}$), whose correlation dimension is $D = \log 2/\log 3 \sim 0.63$. Although the curve seems to follow a general drift, the linear regression is very bad ($\rho = 0.531$) and gives anyway a wrong dimension: $A \simeq -3.72 \ll -2D$. Besides, note how the self-similar structure reappears in the triadic blocks $[3^m, 3^{m+1}]$, $m = 1, 2, \ldots$.

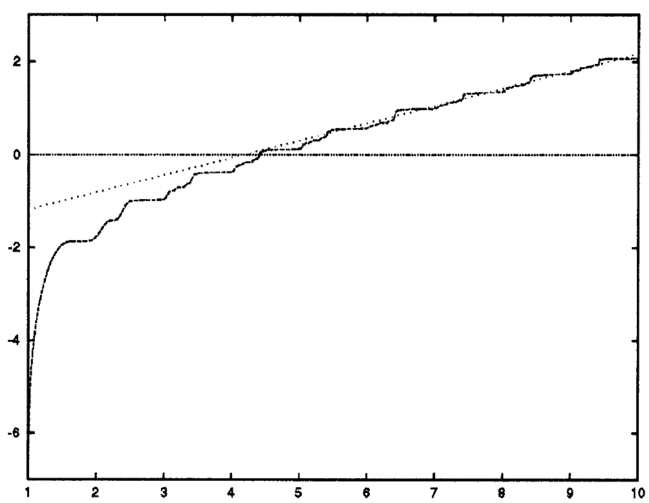

Figure 6. $\log_3(\int_1^p \mathrm{d}p' p'^2 |r(p')|^2)$ against $\log_3(p)$. Once $|r(p)|^2$ has been integrated, it gives the right correlation of the Cantor measure: $A \simeq 0.372 \simeq 1 - D$, with $\rho = 0.992$.

It can be shown that the set U at least contains K. For measures satisfying condition (3.B), the upper and lower correlation dimension coincide ($D^+ = D^- = D$) and are given by the implicit equation (see e.g [GH95b]):

$$\sum_{i=1}^{N} p_i^2 l_i^{-D} = 1. \tag{3.1}$$

This implies, in particular, that $0 < D < 1$. If, furthermore:

Condition 3.C. (Equicontractivity condition.) All the contraction ratios are equal, that is

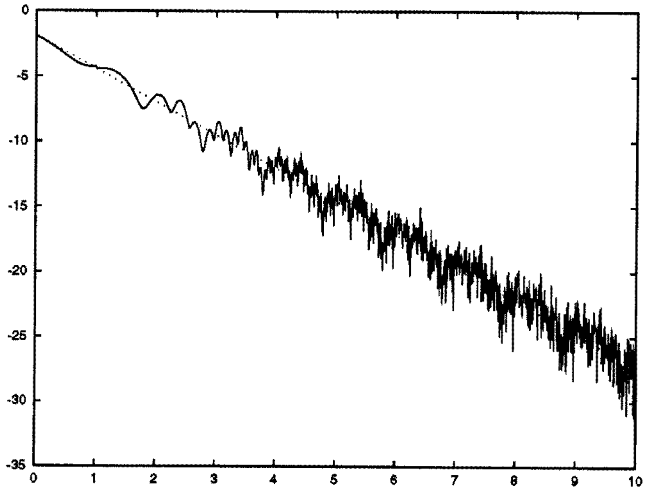

Figure 7. $\log_3 |r(p)|^2$ against $\log_3(p)$ for the triadic Cantor measure with weights $p_1 = \frac{1}{4}$ and $p_2 = \frac{3}{4}$. Again, the linear regression gives a wrong dimension $D \simeq 0.542$ whereas the theoretical dimension is 0.428.

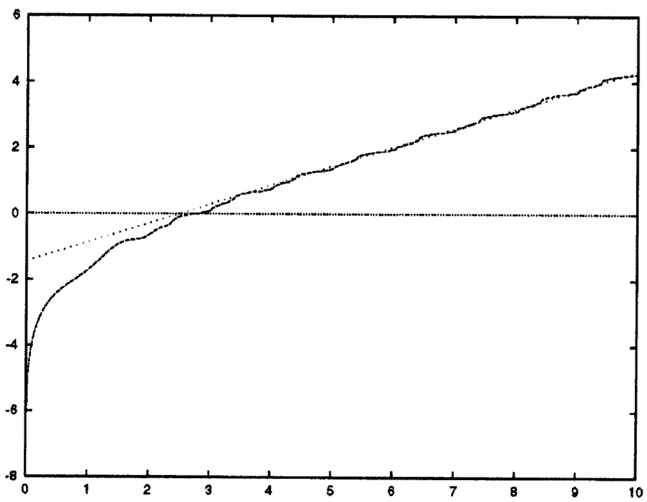

Figure 8. $\log_3(\int_1^p \mathrm{d}p' p'^2 |r(p')|^2)$ against $\log_3(p)$. The linear regression gives an excellent approximation of the correlation dimension: $A \simeq 0.578 \simeq 1 - D$, with $\rho = 0.998$.

$l_i = l, i = l, \ldots, N,$

then D is given explicitly by

$$D = \frac{\log \sum_{i=1}^{N} p_i^2}{\log l}.$$

The case $p_1 = p_2 = \frac{1}{2}$ and $l_1 = l_2 = \frac{1}{3}$ reduces to the well known triadic Cantor measure whose dimension is $D = \log 2 / \log 3$.

The $\mathcal{L}^2$-Fourier asymptotic of self-similar measures has been studied in [Str90], [Str93] and [LW93]. The last authors stated the most general result. They showed (among other

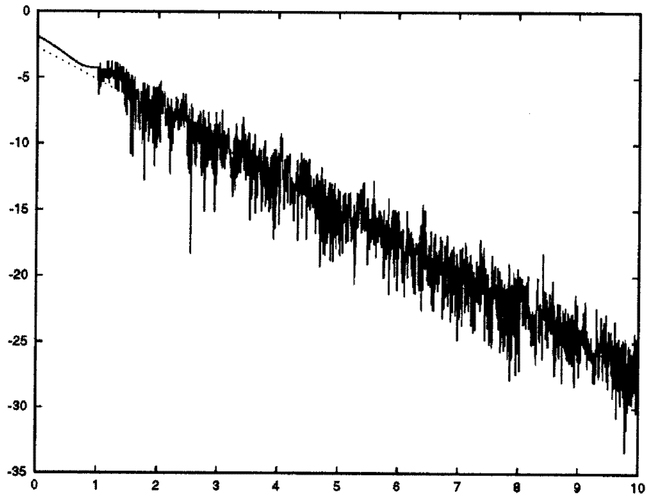

Figure 9. Same as figure 7 but the weights p_1 and p_2 are randomly exchanged at each stage of the construction of the measure with probability a half. The theoretical correlation is unchanged: $D = 0.428$. The experimental dimension ($0.469 = -A - 2$ with $\rho = 0.963$) is still wrong; however, the smoothing effect of randomness makes it closer to the theoretical dimension.

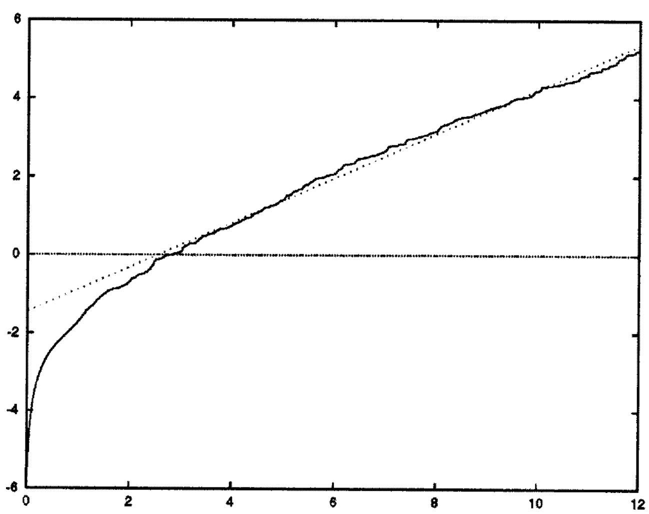

Figure 10. $\log_3(\int_1^p \mathrm{d}p' p'^2 |r(p')|^2)$ against $\log_3(p)$. $A \simeq 0.578 \simeq 1 - D$, with $\rho = 0.997$.

things) that if the strong open set condition holds (the needed condition is actually more general), then

$$\int_{0\leqslant\xi\leqslant R} \mathrm{d}\xi |\hat{V}(\xi)|^2 \sim R^{1-D} \tag{3.2}$$

where the notation $f \sim g$ means $Cf \leqslant g \leqslant C'f$ for some $C' \geqslant C > 0$. An immediate consequence is:

Proposition 3.1. If V satisfies conditions 3.A, 3.B and 3.C, then

$$\int_{p_0}^{p} \mathrm{d}p'\, p'^2 |r(p')|^2 \sim p^{1-D}$$

where D is given by (3.1).

We wish now to show that the self-similar nature of the scatterer can be recovered in the scattering data. As was first shown in [Hol94b] in the case of the triadic Cantor measure, the self-similarity of measures allows a renormalization procedure for their Fourier transform. Let us introduce the following definition. A tempered distribution $\eta \in \mathcal{S}'(\mathbb{R})$ is said to have a large-scale renormalization if there exists $\beta \in \mathbb{C}, \alpha > 1$ and a non-zero distribution $\eta^* \in \mathcal{S}'(\mathbb{R})$ such that

$$w^* - \lim_{n\to\infty} \beta^n \eta(\alpha^n \cdot) = \eta^*$$

the convergence taking place in the weak-$*$ topology of $\mathcal{S}'(\mathbb{R})$. Conversely, a distribution η is said to have a small-scale renormalization around x_0 if for $\beta \in \mathbb{C}, \alpha < 1$ and $\eta^* \neq 0 \in \mathcal{S}'$

$$w^* - \lim_{n\to\infty} \beta^n \eta(\alpha^n(\cdot - x_0) + x_0) = \eta^*.$$

In order to obtain a result concerning the renormalization of the scattering amplitude we have to suppose that the following condition holds for V.

Condition 3.D. $V * \tilde{V}$ satisfies the strong open set condition, where $\tilde{V}(x) = V(-x)$.

We have the following result.

Theorem 3.1. Let V be a self-similar measure satisfying conditions 3.A and 3.B. Then V has a small-scale renormalization around each fixed point $x_i = -b_i/l_i$ of the similarities S_i. Consequently, $\mathrm{e}^{\mathrm{i}x_i\xi}\hat{V}(\xi)$ has a large-scale renormalization. If, furthermore, conditions 3.C and 3.D hold, then $|\hat{V}|^2$ has a large-scale renormalization. Precisely, we have that $(\sum_{i=1}^{N} p_i^2)^{-n}|\hat{V}(l^{-n}\cdot)|^2$ converges in $\mathcal{S}'(\mathbb{R})$.

Proof. The measure V satisfies, in the sense of the distributions,

$$V(x) = \sum_{i=1}^{N} \frac{p_i}{l_i} V \circ S_i^{-1}(x).$$

If we fix some S_i, we have

$$\frac{l_i}{p_i} V \circ S_i(x) - V(x) = \sum_{j\neq i} \frac{l_i p_j}{l_j p_i} V \circ S_j^{-1} \circ S_i(x)$$

and consequently

$$\left(\frac{l_i}{p_i}\right)^{n+1} V \circ S_i^{n+1} - \left(\frac{l_i}{p_i}\right)^{n} V \circ S_i^{n} = \sum_{j\neq 1} \frac{l_i^{n+1} p_j}{l_j p_i^{n+1}} V \circ S_j^{-1} \circ S_i^{n+1}(x)$$

for all integer n. Now the strong open set condition implies that $S_i^{-1} \circ S_j(K) \cap K = \emptyset$ for all $i \neq j$, where K is the support of V. Thus $S_i^{-n-1} \circ S_j(K) \cap S_i^{-n}(K) = \emptyset$ for all integer n, that is

$$\operatorname{supp}\left(\left(\frac{l_i}{p_i}\right)^{n+1} V \circ S_i^{n+1} - \left(\frac{l_i}{p_i}\right)^{n} V \circ S_i^{n}\right) \cap S_i^{-n}(K) = \emptyset. \tag{3.3}$$

Now take some φ in $\mathcal{S}(\mathbb{R})$ and define

$$a_n = \left(\frac{l_i}{p_i}\right)^n \int \mathrm{d}x\, V \circ S_i^n(x)\varphi(x).$$

Let us show that a_n is a converging sequence. Clearly, we have

$$a_{n+1} = \left(\frac{l_i}{p_i}\right)^{n+1} \left\{ \int_{x \in S_i^{-n}(K)} + \int_{x \in S_i^{-n-1}(K)\setminus S_i^{-n}(K)} \right\} \mathrm{d}x\, V \circ S_i^{n+1}(x)\varphi(x)$$

that is, in view of (3.3),

$$a_{n+1} - a_n = \left(\frac{l_i}{p_i}\right)^{n+1} \int_{x \in S_i^{-n-1}(K)\setminus S_i^{-n}(K)} \mathrm{d}x\, V \circ S_i^{n+1}(x)\varphi(x). \tag{3.4}$$

At this point, suppose φ is of compact support. Then we can find some integer N, such that for all $n \geqslant N$, $x \in S_i^{-n-1}(K)\setminus S_i^{-n}(K) \cap \mathrm{supp}(\varphi) = \emptyset$, in which case the right-hand side vanishes and $a_{n+1} = a_n$. This means that $(l_i/p_i)^n V \circ S_i^n$ converges in $\mathcal{D}'(\mathbb{R}) = C_0^{\infty'}(\mathbb{R})$ towards some distribution V_i^*. Now the primitive of V_i^* is polynomially bounded. Indeed, for any $R > 0$ and any integer n such that $\mathrm{supp}(V_i^*) \cap [-R, +R] \subset S_i^{-n}(K)$, we have

$$\int_{|x|\leqslant R} V_i^*(x)\,\mathrm{d}x \leqslant \int_{x\in K} \mathrm{d}x\, V(x) + \int_{x \in S_i^{-1}(K)} \mathrm{d}x\, \frac{l_i}{p_i} V \circ S_i(x) + \cdots$$
$$\cdots + \int_{x \in S_i^{-n}(K)} \mathrm{d}x\, \left(\frac{l_i}{p_i}\right)^n V \circ S_i^n(x) = 1 + p_i^{-1} + \cdots + p_i^{-n}.$$

Since $n \sim -\log R/\log l_i$, this yields

$$\left| \int_{|x|\leqslant R} V_i^*(x)\,\mathrm{d}x \right| \leqslant O\, R^{\log p_i/\log l_i}.$$

Consequently, V_i^* is a tempered distribution. It follows that the convergence of $(l_i/p_i)^n V \circ S_i^n$ can be extended to $\mathcal{S}'(\mathbb{R})$ because if $\varphi \in \mathcal{S}(\mathbb{R})$, then

$$a_{n+1} - a_n \leqslant \sup_{x \in S_i^{-n-1}(K)\setminus S_i^{-n}(K)} \int_{x \in S_i^{-n-1}(K)} \mathrm{d}x\, V_i^*(x) \leqslant O(l_i^{nm}) \qquad \forall m \in \mathbb{N}$$

and (a_n) is a Cauchy, therefore converging, sequence in $\mathbb{C}$.

Since $S_i^n = l_i^n(x - x_i) + x_i$, this means that V has a small-scale renormalization around x_i. We only need to take the Fourier transform (in the sense of distributions) to obtain a renormalization for $\hat{V}$:

$$p_i^{-n} \mathrm{e}^{\mathrm{i} l_i^{-n} x_i \xi}\, \hat{V}(l_i^{-n}\xi) \to \mathrm{e}^{\mathrm{i} x_i \xi}\, \hat{V}_i^*(\xi).$$

Now for the second part of the proof, it is easy to see that if V is equicontractive, then $V * \tilde{V}$ is still self-similar with

$$V * \tilde{V} = \sum_{i \neq j} p_i p_j V * \tilde{V} \circ S_{i,j}^{-1} + \sigma V * \tilde{V} \circ S_0^{-1}$$

with $S_{i,j}(x) = lx + b_i - b_j$, $S_0(x) = lx$, and $\sigma = \sum p_i^2$. If condition 3.D holds, then $V * \tilde{V}$ has a small-scale renormalization around 0, the fixed point of S_0,

$$w^*. \lim_{n\to+\infty} \left(\frac{l}{\sigma}\right)^n (V * \tilde{V})(l^n\cdot) = \eta$$

for some η in S'. Passing to the Fourier transform gives, for all φ in $\mathcal{S}$,

$$\lim_{n\to+\infty} \sigma^n |\hat{V}(l^{-n}\cdot)|^2(\varphi) = \hat{\eta}(\varphi) \tag{3.5}$$

that is $|\hat{V}|^2$ has a large-scale renormalization. □

We wish to apply this result to $|r(p)|^2 \sim |\hat{V}(2p)|^2/p^2$. However, the space of large-scale renormalizable functions is not stable under multiplication by homogeneous functions $p^\lambda, \lambda \leqslant -1$. To remedy this problem, we restrict the space of test functions. We define $\mathcal{S}_{\text{pos}}(\mathbb{R})$ the subset of $\mathcal{S}(\mathbb{R})$ of functions with support in $\mathbb{R}^+$ only and we consider the renormalization in $\mathcal{S}'_{\text{pos}}(\mathbb{R})$ rather than $\mathcal{S}'(\mathbb{R})$. Since $r(p)$ is *a priori* not entirely defined on $\mathbb{R}^+$, we extend it by setting $r(p) = 0$ if $p \leqslant \|V\|/2$. Then, we have:

Proposition 3.2. Assume conditions 3.A, 3.B, 3.C and 3.D to hold. Then $|r(p)|^2$ has a large-scale renormalization in $\mathcal{S}'_{\text{pos}}(\mathbb{R})$. Precisely, we have that $(l^2 \sum_{i=1}^N p_i^2)^{-n}|r(l^{-n}\cdot)|^2$ converges in $\mathcal{S}'(\mathbb{R})$.

Proof. Take some function φ in $\mathcal{S}_{\text{pos}}(\mathbb{R})$. By (2.7) we have

$$\int \mathrm{d}p\, l^{-2n}\sigma^{-n}|r(l^{-n}p)|^2\varphi(p)$$
$$= \sigma^{-n}\int \mathrm{d}p\, \frac{|\hat{V}(l^{-n}p)|^2}{p^2}\varphi(p) + l^{-2n}\sigma^{-n}\int \mathrm{d}p\, \varphi(p)\epsilon(l^{-n}p)$$

where $|\epsilon(p)| \leqslant O(1/p^3)$ is the error term. Let us call I_n and J_n the first and second terms of the right-hand side, respectively. By (3.5), I_n converges towards $\hat{\eta}(\varphi/p^2)$. On the other hand, we have

$$|J_n| \leqslant l^n\sigma^{-n} O\left(\int \mathrm{d}p\left|\frac{\varphi(p)}{p^3}\right|\right).$$

Now $\sigma = \sum_{i=1}^N pi^2 \geqslant \sum_{i=1}^N N^{-2} = N^{-1}$, whereas the strong open set condition imposes $l < N^{-1}$. Thus $l\sigma^{-1} < 1$ and $J_n \to 0$. Hence,

$$(l^2\sigma)^{-n}|r(l^{-n}\cdot)|^2(\varphi) \to R^*(\varphi)$$

where $R^*(\varphi) = \hat{\eta}(\varphi/p^2)$. This completes the proof. □

4. A numerical procedure

We now wish to illustrate some of the above results with a numerical experiment in the basic case of self-similar potentials. In particular, we want to insist on the fact that the correlation dimension of the potential can only appear via an integration of the scattered intensity. To be more precise, suppose the potential is modelled by the triadic Cantor measure V_{c}, Its Fourier transform is given by the infinite product

$$\hat{V}_{\text{c}}(p) = \prod_{j=0}^{\infty} \cos(3^{-j}\pi p).$$

Now $|\hat{V}_{\text{c}}|^2$ has a local maximum $|\hat{V}_{\text{c}}(\pi)|^2$ at every point $p = 3^n\pi$ and a local minimum 0 at every point $p = 3^n(\pi/2)n \in \mathbb{N}$. Consequently, $\limsup(\log|r(p)|^2/\log(p)) = -2$ and $\limsup(\log|r(p)|^2/\log(p)) = -\infty$, that is only the trivial behaviour of $|r(p)|^2$ may appear.

We are going to describe a numerical procedure to compute the exact reflected amplitude on self-similar potentials. Since these latter are ideal mathematical objects, we can only approach them by constructing 'finite' fractals, that is measures involving a finite number of iterations. More precisely, consider a self-similar measure V satisfying the strong open set condition and

$$V(x) = \sum_{i=1}^{N} \frac{p_i}{l_i} V \circ S_i^{-1}(x) \tag{4.1}$$

for a set of weights $\{p_i\}$ and similarities $S_i(x) = l_i(x - x_i) + x_i$, with $x_1 = 0 < x_2 .. < x_N = 1$. Now take χ the characteristic function of the unit interval and form the nth order approximation:

$$V_0 = \chi$$
$$V_n(x) = \Sigma^N_{i_1,\ldots,i_n=1} \frac{p_{i_1}}{l_{i_1}} \cdots \frac{p_{i_n}}{l_{i_n}} \chi \circ S^{-1}_{i_n} \circ \cdots \circ S^{-1}_{i_1}(x).$$

It can be shown that $w - \lim V_n = V$, $n \to \infty$, and $\|V_n\| = \|V\| = 1$. Thus, in view of section 2, $r_{V_n}(p) \to r_V(p)$.

Now, since the approximated measures V_n are piecewise constant functions, we can compute $r_{V_n}(p)$ with the method of transfer matrices. Each transfer matrix $\mathcal{M}(V_n, p)$ can be decomposed into a product of N^n elementary transfer matrices corresponding to each square part of the potential. Rather than computing N^n matrices, we shall here make use of the self-similarity of the potential. Looking at the self-similarity equation (4.1), we see that an nth-order approximation of V is a sum of N disjoint measures being the $(n-1)$th-order approximations of $(p_i/l_i)V \circ S_i^{-1}$, $i = 1, \ldots, N$. So we have, with obvious notation,

$$\mathcal{M}(V_n, p) = \mathcal{M}\left(\frac{p_N}{l_N}(V \circ S_i^{-1})_{n-1,}p\right) \ldots \mathcal{M}\left(\frac{p_1}{l_1}(V \circ S_i^{-1})_{n-1}, p\right)$$

the transfer matrix on the zero-order approximations being known analytically. Thus, for each n, the transfer matrix $\mathcal{M}(V_n, p)$ can easily be computed by a recursive procedure. This gives a numerical expression for $r_{V_n}(p)$. Now let us retrieve the correlation dimension via the reflection amplitude. Therefore, we need to estimate the error induced by an nth-order approximation. It has been shown in [Gue95] that

$$||r_{V_n}(p)|^2 - |r(p)|^2| \leqslant \frac{4l^n}{p} + O\left(\frac{1}{p^3}\right). \tag{4.2}$$

This yields

$$\left| \int_1^p \mathrm{d}p' \, p'^2 |r_{V_n}(p')|^2 - \int_1^p \mathrm{d}p' \, p'^2 |r_V(p')|^2 \right| \leqslant O(l^n p^2) + O(1).$$

On the other hand, we know from (3.2) that $\int_1^p \mathrm{d}p' \, p'^2 |r_V(p')|^2$ is of order p^{1-D}. Thus, taking $l^n \ll p^{-2}$, we are ensured that the relative error we commit is negligible. In figures 5–8 we show an application of this method to the triadic Cantor measure. A comparison is drawn with random self-similar measures. On each figure, we have made a linear regression $y = Ax + B$ and indicated the coefficient of correlation ρ.

Acknowledgment

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